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PROPAGATION ALONG A HELICAL WIRE

by

WILLIAM SOLLFREY

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ABSTRACT

Previous researches on the helix have made simplifying approximations about the nature of the boundary conditions. In this paper the field equations and boundary conditions are formulated exactly. They are then solved by an expansion in powers of the ratio of the thickness of the wire to the distance between turns.

The method used consists of introducing a new coordinate system, which is such that a helical wire of circular cross-section is a surface in which one coordinate is constant. Maxwell's equations and the electromagnetic boundary conditions are expressed in terms of this system. Since non-orthogonal coordinates are involved, the equations cannot be solved exactly, but a perturbation procedure may be applied as indicated above.

The result of the analysis is to show that there is a principal mode, which propagates with the free space velocity of light in the direction of the wire. The characteristics of this mode are studied, and they are compared successfully with experiment.

1. Introduction.

The problem of the electromagnetic modes of a helix has been treated by making idealizations of the physical picture. Thus, the actual helical wire has been idealized to a cylindrical sheath conducting in a fixed direction along its surface, and alternatively to a helical wire of zero thickness. These idealizations entail either physical or mathematical difficulties. In this paper the problem of a helical wire of finite circular cross-section is treated directly in terms of a helical coordinate system. The differential equations so obtained (see below equations 3.21) are solved approximately, yielding a mode which propagates with the velocity of light along the direction of the wire. It may be noted that the coordinate system employed is not orthogonal. The fields of a helical sheath have been discussed by, among others, Kompfner[1], Pierce[2], Chu and Jackson[3], Brillouin[4], and Phillips and Malin[5]. The fields of an infinitely thin helical wire have been discussed by Roubine[6], Parzen[7], and Phillips[8], and in an early paper by Pochlington[9].

It has been generally realized that the theory of the helical sheath is inconsistent with the physical picture of an actual helix. Thus, the fields which are obtained depend on the angle θ, in cylindrical coordinates, as $e^{in\theta}$ which means that the surfaces of constant phase are stationary in space as we move in the axial direction. However, the fields should be expected to turn at the same rate as the helix. Further, the boundary conditions at the sheath are satisfied for all θ, while they should only be satisfied at the wire itself. Thus, the sheath corresponds not to a single helix but to a set of adjacent helices, filling up the cylindrical surface.

The theory of the infinitely thin wire avoids this difficulty but leads to another. To push a finite current thru an infinitely thin wire requires an infinite e.m.f. Hence the tangential component of electric field, calculated by means of the retarded potential, becomes infinite at the wire, while it is actually required to be zero by the boundary conditions. Since the boundary conditions at the wire are necessary to determine the propagation constant of the fields, the theory is inherently inconsistent. The difficulty can be avoided

by impressing different boundary conditions, but such a procedure is open to grave physical objections.

The method presented in this paper avoids these shortcomings by obtaining the fields of a wire of small but finite thickness. The boundary conditions correspond to perfect conductivity at the surface of the wire and the theory is entirely consistent with the physical picture.

The details of the method are as follows. A special coordinate system is introduced, such that the surface of a helical wire of circular cross-section is a surface on which one of the coordinates is constant. Maxwell's equations are expressed in these coordinates, and the appropriate boundary conditions impressed. The principal mode is then calculated in detail, by expansion of the fields in powers of the ratio of the wire thickness to the distance between turns. The propagation constant may be found from the boundary conditions at the wire, and the surface current may be determined.

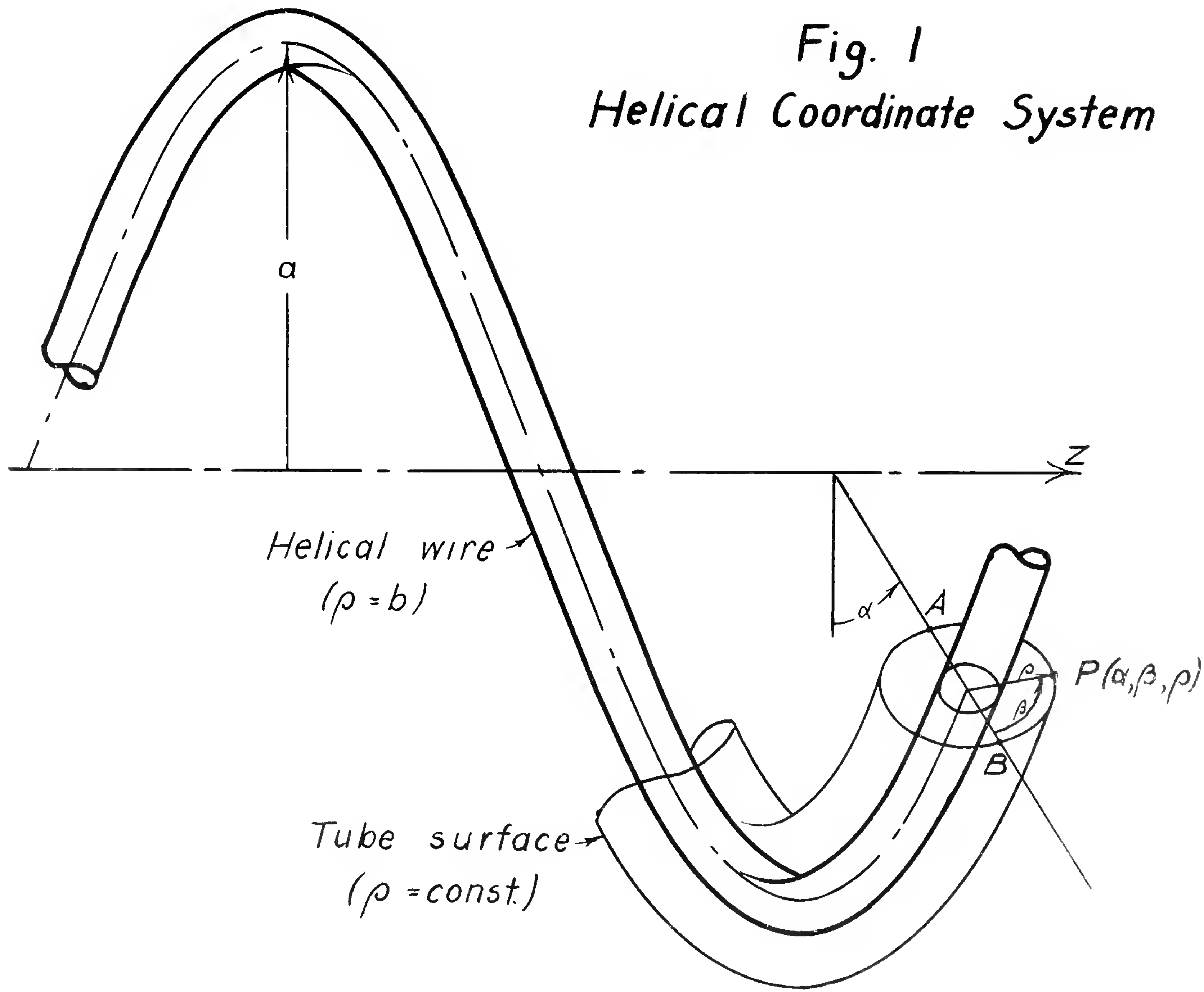
Once the surface current has been found, the fields away from the wire may be obtained by use of retarded potentials. The resulting formulas are quite complicated, but may be separated into terms which are interpreted as circulating waves, that is, the surfaces of constant phase turn as the waves travel. The method of integration used here is somewhat different from that of Roubine[6] or Phillips[8].

We obtain the result that there is a principal mode which propagates with the velocity of light along the direction of the wire under the restrictions that the thickness of the wire is small compared to both the wavelength and the separation between turns. The field structure is given and in particular the value of the longitudinal field on the axis is determined.

The method and results are substantially those presented by the author at the Symposium on Traveling Wave Tubes held in New York on October 26, 1948 under the auspices of the Panel on Electron Tubes of the Research and Development Board.

Fig. 1

Helical Coordinate System



2. Geometry of the Helix

We wish to determine a set of coordinates such that a helical wire of circular cross-section will be a surface on which one coordinate is constant. For this purpose we shall use the following three coordinates, as shown in the accompanying diagram.

First, the arc length along the line of centers of the wire cross-section denoted by α and measured in such units that α increased by 2π in traversing a single turn. Second, consider the set of tube surfaces concentric with the wire. Denote by ρ the radius of the circular cross-section of a surface, measured from the line of centers. A tube surface is therefore determined by an equation $\rho =$ constant, and for the surface of the helical wire itself this constant is equal to b, the radius of the helical wire. The radius of the cylinder on which the line of centers lies will be called a.

As the third coordinate we choose an angle β in the cross-sectional plane of the tube surfaces perpendicular to the line of centers of the wire measured from a reference line determined as follows: The radius vector drawn from the central axis of the cylinder to a point on the line of centers is also perpendicular to the line of centers and hence lies in the cross-sectional plane. This radius vector will intersect the tube surface in two points A and B which may be referred to as the inner and outer points respectively. Now we shall measure the angle β from the line connecting the center point of the circular cross-section of the wire to the outer intersection point.

The three coordinates so defined completely specify a point on any tube surface. These coordinates are single valued only if the radius ρ is sufficiently small. If ρ is increased to too great a value, the tube surface around one turn of the line of centers will intersect that around the next turn, and the angles α, β are no longer unique. This restriction is implied in all subsequent calculations.

We shall now derive the equations connecting the ordinary rectangular coordinates x,y,z with the helical coordinates α, β, ρ. Denote a point on the line of centers by x_0, y_0, z_0. The pitch angle of the helix will be called ψ, measured so that $\psi = \frac{\pi}{2}$ for a straight wire.

We now have:

$$x_o = a \cos \alpha \qquad (2.1)$$
$$y_o = a \sin \alpha$$
$$z_o = a \alpha \tan \psi$$

We shall define three auxiliary vectors, $\underline{i}_r$, $\underline{i}_t$, and $\underline{i}_u$; $\underline{i}_r$ will be the radius vector from the central axis to the line of centers. By its definition, α measures the angular position of a point on the line of centers, except that α ranges from $-\infty$ to ∞ . Hence, as far as trigonometric functions are concerned, $\alpha = \theta$, where θ is the polar angle in cylindrical coordinates with the z axis along the central axis. We thus have

$$\underline{i}_r = \underline{i}_x \cos\alpha + \underline{i}_y \sin\alpha \qquad (2.2)$$

We take $\underline{i}_t$ to be a vector tangential to the line of centers. From the definition of a helix as a line making a constant pitch angle with the z-axis, we have:

$$\underline{i}_t = \underline{i}_\theta \cos\psi + \underline{i}_t \sin\psi \qquad (2.3)$$
$$= -\underline{i}_x \sin\alpha \cos\psi + \underline{i}_y \cos\alpha \cos\psi + \underline{i}_z \sin\psi$$

We define a third vector $\underline{i}_u$ by the relation $\underline{i}_u = \underline{i}_t \times \underline{i}_r$. We may regard $\underline{i}_r$, $\underline{i}_u$, and $\underline{i}_t$ as a set of moving axes with origin at a point on the line of centers. They behave as base vectors of a rectangular coordinate system. The vector $\underline{i}_u$ is given by its definition as:

$$\underline{i}_u = -\underline{i}_x \sin\alpha \sin\psi + \underline{i}_y \cos\alpha \sin\psi - \underline{i}_z \cos\psi \qquad (2.4)$$

The vector distance from the line of centers to a point on the tube surface is given by $\rho\, \underline{i}_\rho$, where $\underline{i}_\rho$ is a unit vector in the direction from the center point to the point on the tube surface. Resolving $\underline{i}_\rho$ into its components gives, using the definition of β ,

$$\underline{i}_\rho = \underline{i}_r \cos\beta + \underline{i}_u \sin\beta \qquad (2.5)$$
$$= \underline{i}_x (\cos\alpha \cos\beta - \sin\alpha \sin\beta \sin\psi)$$
$$+ \underline{i}_y (\sin\alpha \cos\beta + \cos\alpha \sin\beta \sin\psi)$$
$$- \underline{i}_z \sin\beta \cos\psi$$

The vector joining the origin of the basic rectangular coordinate system x,y,z to the point on the wire is given by

(2.6) $$\underline{R} = \underline{i_x}\, x + \underline{i_y}\, y + \underline{i_z}\, z$$

It also is equal to:

(2.7) $$\underline{R} = \underline{i_x}\, x_o + \underline{i_y}\, y_o + \underline{i_z}\, z_o + \underline{i_\rho}\rho$$

since we may resolve $\underline{R}$ into the vector from the origin to the line of centers plus the vector from the line of centers to the tube surface. Equating these two expressions and resolving into components gives:

(2.8) $$\begin{aligned} x &= a\cos\alpha + \rho(\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\psi) \\ y &= a\sin\alpha + \rho(\sin\alpha\cos\beta + \cos\alpha\sin\beta\sin\psi) \\ z &= a\alpha\tan\psi - \rho\sin\beta\cos\psi \end{aligned}$$

These are the desired equations of transformation.

3. Maxwell's Equations in Helical Coordinates

To transform Maxwell's equations into the helical system as described above we use the notation and formalism of Stratton*. The first step is to calculate the unitary vectors. They are defined by:

(3.1) $$\underline{a_i} = \frac{\partial}{\partial u^i}\,\underline{r}$$

Here u^1, u^2, u^3 denote α, β, ρ in that order; $\underline{R} = (x,y,z)$. The first derivatives are given by the table:

(3.2)

a) $x_\alpha = - a\sin\alpha - \rho(\sin\alpha\cos\beta + \cos\alpha\sin\beta\sin\psi)$

b) $y_\alpha = a\cos\alpha + \rho(\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\psi)$

c) $z_\alpha = a\tan\psi$

d) $x_\beta = -\rho(\cos\alpha\sin\beta + \sin\alpha\cos\beta\sin\psi)$

e) $y_\beta = -\rho(\sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\psi)$

f) $z_\beta = -\rho\cos\beta\cos\psi$

g) $x_\rho = \cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\psi$

h) $y_\rho = \sin\alpha\cos\beta + \cos\alpha\sin\beta\sin\psi$

i) $z_\rho = -\sin\beta\cos\psi$

* J. A. Stratton, "Electromagnetic Theory", pp. 38–47.

The vector $\underline{a_1}$ is given as $\underline{i_x}\frac{\partial x}{\partial \alpha} + \underline{i_y}\frac{\partial y}{\partial \alpha} + \underline{i_z}\frac{\partial z}{\partial \alpha}$, and similar formula for $\underline{a_2}$ and $\underline{a_3}$.

We now have for the unitary vectors:

(3.3)

a) $$\underline{a_1} = -\underline{i_x}\left[a \sin\alpha + \rho(\sin\alpha\cos\beta + \cos\alpha\sin\beta\sin\psi)\right]$$
$$+\underline{i_y}\left[a\cos\alpha + \rho(\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\psi)\right]$$
$$+\underline{i_z}\, a\tan\psi$$

b) $$\underline{a_2} = -\underline{i_x}\,\rho(\cos\alpha\sin\beta + \sin\alpha\cos\beta\sin\psi)$$
$$-\underline{i_y}\,\rho(\sin\alpha\sin\rho - \cos\alpha\cos\beta\sin\psi)$$
$$-\underline{i_z}\,\rho\cos\beta\cos\psi$$

c) $$\underline{a_3} = \underline{i_x}(\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\psi)$$
$$+\underline{i_y}(\sin\alpha\cos\beta + \cos\alpha\sin\beta\sin\psi)$$
$$-\underline{i_z}\sin\beta\cos\psi$$

It is somewhat more convenient to express the transformation in terms of the unit vectors of the associated cylindrical coordinate system rather than the unit vectors of the rectangular system. We thus have:

(3.4)

a) $$\underline{i_x} = \underline{i_r}\cos\theta - i_\theta\sin\theta$$

b) $$i_y = \underline{i_r}\sin\theta + i_\theta\cos\theta$$

(3.5)

a) $$r^2 = a^2 + 2a\rho\cos\beta + \rho^2(\cos^2\beta + \sin^2\beta\sin^2\psi)$$

b) $$\theta = \alpha + \tan^{-1}\frac{\rho\sin\beta\sin\psi}{a+\rho\cos\beta}$$

When we insert (3.4) and (3.5) into (3.3) the angles combine so that the only ones

appearing are $\theta - \alpha$ and β. We may simplify by the formulas:

$$(3.6)\quad \begin{aligned} &\text{a)}\quad \cos(\theta-\alpha) = \frac{a+\rho\cos\beta}{r} \\ &\text{b)}\quad \sin(\theta-\alpha) = \frac{\rho\sin\beta\,\sin\psi}{r} \end{aligned}$$

The result is that α no longer appears explicitly. After these simplifications, we have:

(3.7) a) $\underline{a_1} = \underline{i_\theta}\, r + \underline{i_z}\, a \tan\psi$

b) $\underline{a_2} = -\underline{i_r}\,\frac{\rho}{r}\sin\beta\,(a+\rho\cos\beta\cos^2\psi)$

$\qquad +i_\theta\,\frac{\rho}{r}\sin\psi(a\cos\beta+\rho)$

$\qquad -\underline{i_z}\,\rho\cos\beta\cos\psi$

c) $\underline{a_3} = \underline{i_r}\,\frac{1}{r}\left[a\cos\beta+\rho(\cos^2\beta+\sin^2\beta\,\sin^2\psi)\right]$

$\qquad +i_\theta\,\frac{a}{r}\sin\beta\sin\psi - \underline{i_z}\sin\beta\cos\psi$

We may find the line elements without much difficulty. They are given by $g_{ij} = \underline{a_i}\cdot\underline{a_j}$. We have:

(3.8) a) $g_{11} = r^2 + a^2\tan^2\psi$

b) $g_{12} = \rho^2\sin\psi$

c) $g_{13} = 0$

d) $g_{21} = \rho^2\sin\psi$

e) $g_{22} = \rho^2$

f) $g_{23} = 0$

g) $g_{31} = 0$

h) $g_{32} = 0$

i) $g_{33} = 1$

It will be noted that since $g_{21} = g_{12} = \underline{a}_1 \cdot \underline{a}_2$ are not zero the unitary vectors are not orthogonal.

The unit vectors are therefore given by:

(3.9) a) $$\underline{i}_1 = \frac{1}{\sqrt{g_{11}}}\left[\underline{i}_\theta\, r + \underline{i}_z\, a \tan\psi\right]$$

b) $$\underline{i}_2 = -\underline{i}_r \frac{\sin\beta}{r}(a + \rho\cos\beta\cos^2\psi) + \underline{i}_\theta \frac{\sin\psi}{r}(a\cos\beta + \rho) - \underline{i}_z \cos\beta\cos\psi$$

c) $$\underline{i}_3 = \underline{i}_r \frac{1}{r}\left[a\cos\beta + \rho(\cos^2\beta + \sin^2\beta\,\sin^2\psi)\right] + \underline{i}_\theta \frac{1}{r}\, a\sin\beta\sin\psi - \underline{i}_z \sin\beta\cos\psi$$

The volume factor g is given by:

(3.10) $$g = \begin{vmatrix} r^2 + a^2\tan^2\psi & \rho^2\sin\psi & 0 \\ \rho^2\sin\psi & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho^2\left[r^2 + a^2\tan^2\psi - \rho^2\sin^2\psi\right] = \rho^2(a\sec\psi + \rho\cos\beta\cos\psi)^2$$

We may expand an arbitrary vector $\underline{F}$ by

(3.11) $$\underline{F} = \underline{i}_1 F_1 + \underline{i}_2 F_2 + \underline{i}_3 F_3$$

The next step is to find the curl of a vector. This is given by:*

(3.12) $$\nabla \times \underline{F} = \frac{1}{\sqrt{g}}\left[\underline{a}_1\left(\frac{\partial f_3}{\partial u^2} - \frac{\partial f_2}{\partial u^3}\right) + \underline{a}_2\left(\frac{\partial f_1}{\partial u^3} - \frac{\partial f_3}{\partial u^1}\right) + \underline{a}_3\left(\frac{\partial f_2}{\partial u^1} - \frac{\partial f_1}{\partial u^2}\right)\right]$$

Here f_1, f_2, f_3 denote the covariant components of F. We wish to express them in terms of the unit components F_1, F_2, F_3. This is easily done, since we have,**

(3.13) $$F_j = \sqrt{g_{jj}}\; f^j$$

$$f_i = \sum_j g_{ij}\, f^j$$

* Ibid., p. 47, equation 63

** Ibid. p. 41, equation 24; p. 40, equation 19.

Therefore

(3.14) $$f_i = \sum_j \frac{g_{ij}}{\sqrt{g_{jj}}} \; F_j$$

We also have:

(3.15) $$\underline{a}_i = \sqrt{g_{ii}} \; \underline{i}_i$$

Putting these into (3.12) gives:

(3.16) $$\nabla \times \underline{F} = \frac{1}{\sqrt{g}} \left[\underline{i}_1 \sqrt{g_{11}} \sum_j \left(\frac{\partial}{\partial u^2} \frac{g_{3j}}{\sqrt{g_{jj}}} - \frac{\partial}{\partial u^3} \frac{g_{2j}}{\sqrt{g_{jj}}} \right) F_j \right.$$
$$+ \underline{i}_2 \sqrt{g_{22}} \sum_j \left(\frac{\partial}{\partial u^3} \frac{g_{1j}}{\sqrt{g_{jj}}} - \frac{\partial}{\partial u^1} \frac{g_{3j}}{\sqrt{g_{jj}}} \right) F_j$$
$$\left. + \underline{i}_3 \sqrt{g_{33}} \sum_j \left(\frac{\partial}{\partial u^1} \frac{g_{2j}}{\sqrt{g_{jj}}} - \frac{\partial}{\partial u^2} \frac{g_{1j}}{\sqrt{g_{jj}}} \right. \; F_j \right]$$

Formulas (3.11) to (3.16) are valid in any coordinate system. We shall now specialize to the helical coordinates α, β, ρ.

It is convenient to define a set of operators

(3.17) a) $$D_{1j} = \frac{\sqrt{g_{11}}}{\sqrt{g}} \left[\frac{\partial}{\partial u^2} \frac{g_{3j}}{\sqrt{g_{jj}}} - \frac{\partial}{\partial u^3} \frac{g_{2j}}{\sqrt{g_{jj}}} \right]$$

b) $$D_{2j} = \frac{\sqrt{g_{22}}}{\sqrt{g}} \left[\frac{\partial}{\partial u^3} \frac{g_{1j}}{\sqrt{g_{jj}}} - \frac{\partial}{\partial u^1} \frac{g_{3j}}{\sqrt{g_{jj}}} \right]$$

c) $$D_{3j} = \frac{\sqrt{g_{33}}}{\sqrt{g}} \left[\frac{\partial}{\partial u^1} \frac{g_{2j}}{\sqrt{g_{jj}}} - \frac{\partial}{\partial u^2} \frac{g_{1j}}{\sqrt{g_{jj}}} \right]$$

In terms of the D's equation (3.16) becomes

(3.18) $$\nabla \times \underline{F} = \sum_j \left\{ \underline{i}_1 \, D_{1j} + \underline{i}_2 D_{2j} + \underline{i}_3 \, D_{3j} \right\} F_j$$

The explicit forms of the D operators are found to be:

(3.19)

a) $$D_{11} = \frac{\sqrt{g_{11}}}{\sqrt{g}}\left[\frac{\partial}{\partial\beta}\frac{g_{31}}{\sqrt{g_{11}}} - \frac{\partial}{\partial\rho}\frac{g_{21}}{\sqrt{g_{11}}}\right]$$

$$= -\frac{\sqrt{g_{11}}}{\sqrt{g}}\frac{\partial}{\partial\rho}\frac{\rho^2 \sin\psi}{\sqrt{g_{11}}}$$

b) $$D_{12} = \frac{\sqrt{g_{11}}}{\sqrt{g}}\left[\frac{\partial}{\partial\beta}\frac{g_{32}}{\sqrt{g_{22}}} - \frac{\partial}{\partial\rho}\frac{g_{22}}{\sqrt{g_{22}}}\right]$$

$$= -\frac{\sqrt{g_{11}}}{\sqrt{g}}\frac{\partial}{\partial\rho}\rho$$

c) $$D_{13} = \frac{\sqrt{g_{11}}}{\sqrt{g}}\left[\frac{\partial}{\partial\beta}\frac{g_{33}}{\sqrt{g_{33}}} - \frac{\partial}{\partial\rho}\frac{g_{23}}{\sqrt{g_{33}}}\right]$$

$$= \frac{\sqrt{g_{11}}}{\sqrt{g}}\frac{\partial}{\partial\beta}$$

d) $$D_{21} = \frac{\sqrt{g_{22}}}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}\frac{g_{11}}{\sqrt{g_{11}}} - \frac{\partial}{\partial\alpha}\frac{g_{31}}{\sqrt{g_{11}}}\right]$$

$$= \frac{\rho}{\sqrt{g}}\frac{\partial}{\partial\rho}\sqrt{g_{11}}$$

e) $$D_{22} = \frac{\sqrt{g_{22}}}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}\frac{g_{12}}{\sqrt{g_{22}}} - \frac{\partial}{\partial\alpha}\frac{g_{32}}{\sqrt{g_{22}}}\right]$$

$$= \frac{\rho}{\sqrt{g}}\frac{\partial}{\partial\rho}\rho\sin\psi$$

f) $$D_{23} = \frac{\sqrt{g_{22}}}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}\frac{g_{13}}{\sqrt{g_{33}}} - \frac{\partial}{\partial\alpha}\frac{g_{33}}{\sqrt{g_{33}}}\right]$$

$$= -\frac{\rho}{\sqrt{g}}\frac{\partial}{\partial\alpha}$$

g) $$D_{31} = \frac{\sqrt{g_{33}}}{\sqrt{g}}\left[\frac{\partial}{\partial\alpha}\frac{g_{21}}{\sqrt{g_{11}}} - \frac{\partial}{\partial\beta}\frac{g_{11}}{\sqrt{g_{11}}}\right]$$

$$= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\alpha}\frac{\rho^2\sin\psi}{\sqrt{g_{11}}} - \frac{\partial}{\partial\beta}\sqrt{g_{11}}\right]$$

h) $$D_{32} = \frac{\sqrt{g_{33}}}{\sqrt{g}}\left[\frac{\partial}{\partial\alpha}\frac{g_{22}}{\sqrt{g_{22}}} - \frac{\partial}{\partial\beta}\frac{g_{12}}{\sqrt{g_{22}}}\right]$$

$$= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\alpha}\rho - \frac{\partial}{\partial\beta}\rho\sin\psi\right]$$

i) $$D_{33} = \frac{\sqrt{g_{33}}}{\sqrt{g}}\left[\frac{\partial}{\partial\alpha}\frac{g_{23}}{\sqrt{g_{33}}} - \frac{\partial}{\partial\beta}\frac{g_{13}}{\sqrt{g_{33}}}\right]$$

$$= 0$$

None of the D operators involve α explicitly, but only derivatives with respect to α. We can accordingly separate a factor $e^{i\Gamma\alpha}$ from all components of the electric and magnetic fields, and replace $\frac{\partial}{\partial\alpha}$ by $i\Gamma$. This is entirely to be expected, since the fields should behave like waves traveling along the wire. The Maxwell equations are:

(3.20) a) $\nabla \times \underline{E} = i\omega\mu\underline{H}$

b) $\nabla \times \underline{H} = -i\omega\epsilon\underline{E}$

The time dependence $e^{-i\omega t}$ is implied. Writing $\underline{E} = \underline{E}(\beta,\rho)e^{i\Gamma\alpha}$, $\underline{H} = \underline{H}(\beta,\rho)e^{i\Gamma\alpha}$, the six equations become:

$$\text{(3.21)} \quad \text{a)} \quad i\omega\mu H_\alpha = \frac{\sqrt{g_{11}}}{\sqrt{g}}\left[-\frac{\partial}{\partial\rho}\frac{\rho^2\sin\psi}{\sqrt{g_{11}}}E_\alpha - \frac{\partial}{\partial\rho}\rho E_\beta + \frac{\partial}{\partial\beta}E_\rho\right]$$

$$\text{b)} \quad i\omega\mu H_\beta = \frac{\rho}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}\sqrt{g_{11}}\,E_\alpha + \frac{\partial}{\partial\rho}\rho\sin\psi E_\beta - i\Gamma E_\rho\right]$$

$$\text{c)} \quad i\omega\mu H_\rho = \frac{1}{\sqrt{g}}\left[\left(\frac{i\Gamma\rho^2\sin\psi}{\sqrt{g_{11}}} - \frac{\partial}{\partial\beta}\sqrt{g_{11}}\right)E_\alpha + \rho\left(i\Gamma - \sin\psi\frac{\partial}{\partial\beta}\right)E_\beta\right]$$

$$\text{d)} \quad -i\omega\epsilon E_\alpha = \frac{\sqrt{g_{11}}}{\sqrt{g}}\left[-\frac{\partial}{\partial\rho}\frac{\rho^2\sin\psi}{\sqrt{g_{11}}}H_\alpha - \frac{\partial}{\partial\rho}\rho H_\beta + \frac{\partial}{\partial\beta}H_\rho\right]$$

$$\text{e)} \quad -i\omega\epsilon E_\beta = \frac{\rho}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}\sqrt{g_{11}}\,H_\alpha + \frac{\partial}{\partial\rho}\rho\sin\psi H_\beta - i\Gamma H_\rho\right]$$

$$\text{f)} \quad -i\omega\epsilon E_\rho = \frac{1}{\sqrt{g}}\left[\left(\frac{i\Gamma\rho^2\sin\psi}{\sqrt{g_{11}}} - \frac{\partial}{\partial\beta}\sqrt{g_{11}}\right)H_\alpha + \rho\left(i\Gamma - \sin\psi\frac{\partial}{\partial\beta}\right)H_\beta\right]$$

This is a system of six partial differential equations in six unknown functions. Since there is no impressed external field, we can expect that a field configuration which does not vanish identically can only exist for certain values of the propagation constant Γ, if we assume the frequency ω to be fixed. To each characteristic value of Γ will correspond a mode of transmission.

In addition to the differential equations, it is necessary to impose boundary conditions at the surface of the helical wire. We shall assume the wire to be of infinite conductivity. This implies that the tangential electric and normal magnetic fields vanish at the surface.

Now the vectors $\underline{i_1}$ and $\underline{i_2}$ are tangent to a surface of constant ρ, while $\underline{i_3}$ is normal. Accordingly the boundary conditions are:

$$\left.\begin{aligned} &a)\ E_\alpha = 0 \\ &b)\ E_\beta = 0 \\ &c)\ H_\rho = 0 \end{aligned}\right\} \quad \rho = b\ . \tag{3.22}$$

The differential equations (3.21)and boundary conditions (3.22) constitute an exact formulation of the problem.

4. Solution of the Near Field Equations.

The differential equations (3.21),which give the exact formulation of the fields about the helix, are far too complicated to hope for an explicit solution. Accordingly, we shall seek an approximate solution, based on the consideration that the ratio of the wire radius to the distance between successive turns of the helix is small. This implies that the fields around any one turn are not significantly affected by the fields around adjacent turns. The field around a section of the wire may be considered as composed of a cylindrically symmetric field produced by the currents in this section of the wire and a distortion due to the proximity of adjacent turns. It is the proximity effect which we neglect, and as a first approximation the fields may therefore be taken as independent of the angle β, and derivatives with respect to β may be set equal to zero.

To measure the ratio between the wire radius and the distance between turns we may conveniently employ $b/a \sec\psi$, and then expand the fields in powers of this ratio, dropping powers higher than the first. For a consideration of the fields in the neighborhood of the wire, ρ is of the same order of magnitude as b, and higher powers of $\rho/a \sec\psi$ may similarly be neglected.

With this approximation, the quantities g_{11} and g, given by (3.8a) and (3.10), become:

$$a)\ \sqrt{g_{11}} = a \sec\psi \tag{4.1}$$

$$b)\ \sqrt{g} = a\rho \sec\psi$$

We shall separate the fields into transverse electric and transverse magnetic modes. Here "transverse" means having no component along the wire. We consider first the transverse magnetic case, and set $H_\alpha = 0$.

Maxwell's equations (3.21) become, with these simplifications

$$(4.2)\quad a)\quad 0 = \frac{1}{\rho}\left[-\frac{\partial}{\partial\rho}\frac{\rho^2\sin\psi}{a\sec\psi}E_\alpha - \frac{\partial}{\partial\rho}\rho E_\beta\right]$$

$$b)\quad i\omega\mu H_\beta = \frac{1}{a\sec\psi}\left[\frac{\partial}{\partial\rho}a\sec\psi E_\alpha + \frac{\partial}{\partial\rho}\rho\sin\psi E_\beta - i\Gamma E_\rho\right]$$

$$c)\quad i\omega\mu H_\rho = \frac{1}{a\rho\sec\psi}\left[\frac{i\Gamma\rho^2\sin\psi}{a\sec\psi}E_\alpha + i\Gamma_\rho\ E_\beta\right]$$

$$d)\quad -i\omega\epsilon E_\alpha = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho H_\beta$$

$$e)\quad -i\omega\epsilon E_\beta = \frac{1}{a\sec\psi}\left[\frac{\partial}{\partial\rho}\rho\sin\psi H_\beta - i\Gamma H_\rho\right]$$

$$f)\quad -i\omega\epsilon E_\rho = \frac{1}{a\rho\sec\psi}\, i\Gamma_\rho\ H_\beta$$

We may integrate (4.2a), obtaining:

$$(4.3)\qquad 0 = \frac{\rho^2\sin\psi}{a\sec\psi}E_\alpha + \rho E_\beta$$

$$E_\beta = -\frac{\rho\sin\psi\cos\psi}{a}E_\alpha$$

This result, substituted into (4.2c), yields $H_\rho = 0$. We now have all components of electric field given in terms of H_β by:

$$(4.4)\qquad a)\quad -i\omega\epsilon E_\alpha = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho H_\beta$$

$$b)\quad -i\omega\epsilon E_\beta = \frac{\sin\psi\cos\psi}{a}\frac{\partial}{\partial\rho}\rho H_\beta$$

$$c)\quad -i\omega\epsilon E_\rho = \frac{i\Gamma\cos\psi}{a}H_\beta$$

Multiplying 4.2b) by $-i\omega\epsilon$ and substituting from (4.4) gives:

$$(4.5)\quad k^2 H_\beta = \frac{1}{a\sec\psi}\left[-a\sec\psi\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho H_\beta + \frac{\sin^2\psi\cos\psi}{a}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\rho H_\beta + \frac{\Gamma^2\cos\psi}{a}H_\beta\right]$$

Here $k^2 = \omega^2\mu\epsilon$

We now have on simplifying:

$$(4.6)\quad \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho H_\beta = \frac{\partial}{\partial\rho}\frac{1}{\rho}\left[\rho H_\beta' + H_\beta\right]$$

$$= \frac{\partial}{\partial\rho}\left[H_\beta' + \frac{1}{\rho}H_\beta\right]$$

$$= H_\beta'' + \frac{1}{\rho}H_\beta' - \frac{1}{\rho^2}H_\beta$$

$$(4.7)\quad \frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\rho H_\beta = \frac{\partial}{\partial\rho}\rho\left[\rho H_\beta' + H_\beta\right]$$

$$= \frac{\partial}{\partial\rho}\left[\rho^2 H_\beta' + \rho H_\beta\right]$$

$$= \rho^2 H_\beta'' + 3\rho H_\beta' + H_\beta$$

It follows that the second term in the bracket in (4.5) is of order $(\rho/a\sec\psi)^2$ compared to the first. Consequently, it is to be neglected, since it is of the order of magnitude of already neglected quantities. With this, we have:

$$(4.8)\quad k^2 H_\beta = \left[-H_\beta'' - \frac{1}{\rho}H_\beta' + \frac{1}{\rho^2}H_\beta + \frac{\Gamma^2\cos^2\psi}{a^2}H_\beta\right]$$

$$(4.9)\quad H_\beta'' + \frac{1}{\rho}H_\beta' - \left[\frac{\Gamma^2\cos^2\psi}{a^2} - k^2 + \frac{1}{\rho^2}\right]H_\beta = 0$$

Equation (4.9) is the differential equation for the modified Bessel functions of order one.*

* G. N. Watson, "Theory of Bessel Functions", p. 77.

The boundary conditions (3.23) must also be simplified. By using (4.4) we see that a sufficient boundary condition is:

(4.10) $$\frac{\partial}{\partial\rho}\rho H_\phi = 0 \qquad \rho = b$$

Let us write $u^2 = \frac{\Gamma^2\cos^2\psi}{a^2} - k^2$. Equation (4.9) becomes:

(4.11) $$H_\phi'' + \frac{1}{\rho} H_\phi' - (u^2 + \frac{1}{\rho^2}) H_\phi = 0$$

This equation has the two linearly independent solutions:

(4.12) $$\text{a)} \quad H_\phi = AI_1(u\rho)$$

$$\text{b)} \quad H_\phi = BK_1(u\rho)$$

Let us consider the properties of these two functions. $I_1(u\rho)$ is a monotone increasing function of ρ, behaving like ρ for ρ small. $K_1(u\rho)$ is monotone decreasing, and behaves like $1/\rho$ for small. Hence, since we only consider small values of ρ, we may omit the solution $AI_1(u\rho)$, which, when compared to $BK_1(u.\rho)$, is of the order of magnitude of already neglected terms. Also, physically, the field must decrease as we move away from the wire which again excludes the $AI_1(u\rho)$ solution.

We now must apply the boundary condition (4.10). We have the formula:*

(4.13) $$\frac{d}{dz} zK_1(z) = -zK_0(z)$$

from which it follows that:

(4.14) $$0 = B\, u\, b\, K_0(ub)$$

If the constant B is not to vanish, u must be a root of equation (4.14). Now the principle branch of $K_0(ub)$ has no zeros in the finite portion of the complex u plane**. Hence the only possible root of (4.14) is $u = 0$. Near $u = 0$, $K_0(ub)$ is approximately equal to $-\log 1/2\, ub$, so $u = 0$ actually is a root.

From the definition of u, we accordingly have:

(4.15) $$\frac{\Gamma^2\cos^2\psi}{a^2} - k^2 = 0$$

$$\Gamma = \pm k\, a \sec\psi$$

*Watson, ibid, p. 79, equation 5.

These roots are associated with waves traveling with equal velocities in opposite directions. It will now be shown that they actually propagate with the velocity of light along the wire. This is easily seen. The fields contain as a factor $\exp -i(\omega t - \Gamma\alpha)$ which is equal to $\exp -i\omega(t - \frac{a \sec\psi}{c}\alpha)$. α increases by 2π in going the distance along the wire $2\pi a \sec\psi$, whence we may write, calling s the actual arc length along the wire, $\exp -i\omega(t - s/c)$. This is a wave going at the velocity of light along the wire.

The next step is to determine the actual field components which may be most conveniently expressed in terms of the total current carried by the wire. The surface current density is given by:*

(4.16) $$\underline{J} = -\underline{n} \times \underline{H} .$$

Here $\underline{n}$ is the unit vector normal to the wire, for which we may take $\underline{i_3}$. Hence:

(4.17) $$\underline{J} = (\underline{i_2} \times \underline{i_3})\, H_\beta$$

The vector $(\underline{i_2} \times \underline{i_3})$ is also of unit magnitude and is equal to:

(4.18) $$\underline{i_2} \times \underline{i_3} = \underline{i_r}\,\frac{\rho \sin\beta\, \sin\psi\cos\psi}{r} + \underline{i_\theta}\,\frac{\cos\psi}{r}(a + \rho\cos\beta) + \underline{i_z}\sin\psi$$

To the degree of approximation to which we are working, this becomes:

(4.19) $$\underline{i_2} \times \underline{i_3} = \underline{i_\theta}\cos\psi + \underline{i_z}\sin\psi$$

The total current is obtained by integrating (4.17) around the wire, or what is the same thing, since H_β is independent of β multiplying by $2\pi b$. Hence:

(4.20) $$I = 2\pi b\; BK_1(ub)$$

$$B = \frac{I}{2\pi b}\;\frac{1}{K_1(ub)}$$

$$H_\beta = \frac{I}{2\pi b}\;\frac{K_1(u\rho)}{K_1(ub)}$$

If $u\rho$ is sufficiently small, we may replace $K_1(u\rho)$ by $1/u\rho$, and $K_1(ub)$ by $1/ub$. Then:

(4.21) $$H_\beta = \frac{I}{2\pi\rho}$$

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* Stratton, loc. cit., p. 37.

This expression is the same as that for the field around a straight wire*, as is to be expected, since we have neglected the proximity effect due to the coiling of the wire. The other field components are now easily found from (4.4) and (4.13) as:

(4.22)
$$\text{a)}\ E_\alpha = -\frac{1}{i\omega\epsilon}\frac{I}{2\pi b}\frac{uK_0(u\rho)}{K_1(ub)} \sim \frac{1}{i\omega\epsilon}\frac{I}{2\pi b}u^2 b \log\frac{1}{2}u\rho$$

$$\text{b)}\ E_\beta = \frac{1}{i\omega\epsilon}\frac{I}{2\pi b}\frac{u\,K_0(u\rho)}{K_1(ub)}\frac{\rho}{a}\sin\psi\cos\psi \sim -\frac{1}{i\omega\epsilon}\frac{I}{2\pi b}u^2 b\log\frac{1}{2}u\rho\cdot\frac{\rho}{a}\sin\psi\cos\psi$$

$$\text{c)}\ E_\rho = -\frac{k}{\omega\epsilon}\frac{I}{2\pi b}\frac{K_1(u\rho)}{K_1(ub)} \sim -\sqrt{\frac{\mu}{\epsilon}}\frac{I}{2\pi\rho}$$

From these equations we see that E_α and E_β are negligible off the wire, since u is zero. Actually, because the conductivity is finite, u is not precisely zero, and hence we have retained the forms (4.22) a, b.

This completes the determination of the propagation constant of the principal mode by use of the near fields. We shall next use the current density (4.17) to find the far fields by use of retarded potentials.

5. <u>The Fields off the Wire</u>

To find the fields off the wire, we shall use the retarded potentials, in particular, the retarded Hertz vector given by:**

(5.1)
$$\underline{\Pi}(\underline{R}) = \frac{1}{4\pi\omega\epsilon}\int \underline{J}(\underline{R}')\frac{e^{ik|\underline{R}-\underline{R}'|}}{|\underline{R}-\underline{R}'|}\,dv'$$

Eence we have written $|\underline{R}-\underline{R}'|$ for $\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}$.
If we substitute from (2.8) we have:

(5.2)
$$|\underline{R}-\underline{R}'|^2 = r^2 + a^2 - 2a_r\cos(\theta-\alpha') + (z-a\tan\psi\alpha')^2 + \rho^2$$
$$-2\rho'\left[\left\{r\cos(\theta-\alpha')-a\right\}\cos\beta' + \left\{r\sin(\theta-\alpha')\sin\psi-(z-a\tan\psi\alpha')\cos\psi\right\}\sin\beta'\right]$$

Here $\underline{R}' = (\alpha', \beta', \rho')$ represents a point on the surface of the helix. To the degree of approximation to which we are working, we may drop terms involving

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* Stratton, ibid, p. 535, Equation 52.

**Stratton, ibid, p. 431, Equation 44.

ρ', and obtain:

(5.3) $$\left|\underline{R} - \underline{R}'\right|^2 = r^2 + a^2 - 2a\rho \cos(\theta - \alpha') + (z - a \tan\psi\alpha')^2$$

The current vector $\underline{J}$ is given by:

(5.4) $$\underline{J} = (\underline{i}_{\theta'} \cos\psi + \underline{i}_{z'} \sin\psi)\, \frac{I}{2\pi b}\, e^{i\Gamma\alpha'}$$

In order to perform the integrations, we wish to eliminate the dependence of the vectors $\underline{i}_{\theta'}$ and $\underline{i}_{z'}$ on the coordinate α'. This may be done as follows:

(5.3) $$\underline{i}_{z'} = \underline{i}_z$$

$$\underline{i}_{\theta'} = -\underline{i}_x \sin\theta' + \underline{i}_y \cos\theta'$$

$$= -(\underline{i}_r \cos\theta - \underline{i}_\theta \sin\theta)\sin\alpha' + (\underline{i}_r \sin\theta + \underline{i}_\theta \cos\theta)\cos\alpha'$$

$$= \underline{i}_r \sin(\theta - \alpha') + \underline{i}_\theta \cos(\theta - \alpha')$$

The volume element becomes the surface element $2\pi b\, a \sec\psi\, d\alpha'$, whence the integral reduces to:

(5.6) $$\underline{\Pi} = \frac{1}{4\pi\omega\varepsilon}\, I\, a \sec\psi \int_{-\infty}^{\infty} d\alpha' \left[\underline{i}_r \sin(\theta - \alpha')\cos\psi + \underline{i}_\theta \cos(\theta-\alpha')\cos\psi + \underline{i}_z \sin\psi\right] e^{i\Gamma\alpha'} / \left|\underline{R}-\underline{R}'\right|$$
$$\exp ik\left[r^2 + a^2 - 2a\, r\cos(\theta - \alpha') + (z - a\tan\psi\alpha')^2\right]^{1/2}$$

We shall make the change of variables $\alpha' - \dfrac{z}{\alpha\tan\psi} = \alpha$, write $\Gamma = ka\sec\psi$, and write $\theta - \dfrac{z}{a\tan\psi} = \phi$. The integral becomes:

(5.7)
$$\underline{\Pi} = \frac{1}{4\pi\omega\varepsilon}\, I\, a \sec\psi \int_{-\infty}^{\infty} d\alpha \left[\underline{i}_r \sin(\phi - \alpha)\cos\psi + \underline{i}_\theta \cos(\phi - \alpha)\cos\psi + \underline{i}_z \sin\psi\right] e^{ika\sec\psi\left[\alpha + \frac{z}{a\tan\psi}\right]} / \left|\underline{R}-\underline{R}'\right|$$
$$\exp\, ik\left[r^2 + a^2 - 2a\, r\cos(\phi - \alpha) + a^2\tan^2\psi\alpha^2\right]^{1/2}$$

Let us write $\gamma = k\csc\psi$, whence the integral becomes $e^{i\gamma z}$ times a function of ϕ above. Now the surfaces on which ϕ and r are constant are helices,

whence we see that we have a representation in terms of circulating waves, turning at the same rate as the helix.*

The integral may be evaluated by use of the integral representation of Sommerfeld**

$$(5.8)\qquad \frac{e^{ik\left[r^2+z^2\right]^{1/2}}}{\left[r^2+z^2\right]^{1/2}} = \int_0^\infty \frac{\lambda d\lambda}{\sqrt{\lambda^2-k^2}}\, J_o(\lambda r)e^{-\sqrt{\lambda^2-k^2}\,|z|}$$

We insert this into (5.7), and obtain:

$$(5.9)\qquad \underline{\Pi} = \frac{i}{4\pi\omega\epsilon} I\, a \sec\psi\, e^{i\Upsilon z}\int_{-\infty}^{\infty} d\alpha\left[\underline{i_r}\sin(\phi-\alpha)\cos\psi + \underline{i_\theta}\cos(\phi-\alpha)\cos\psi + \underline{i_z}\sin\psi\right)e^{ika\sec\psi\alpha}$$

$$\int_0^\infty \frac{\lambda d\lambda}{\sqrt{\lambda^2-k^2}}\, J_o\left(\lambda\sqrt{r^2+a^2-2a\,r\cos(\phi-\alpha)}\right)\, e^{-\sqrt{\lambda^2-k^2}\;a\tan\psi|\alpha|}$$

The λ integral is over a path indented below at $\lambda = k$. This is equivalent to assuming that k has a small positive imaginary part, which is set equal to zero after the integration. The double integral converges absolutely, and we may change the order of the integrals. The Bessel function, J_o, may be expanded by means of the addition theorem***

$$(5.10)\qquad J_o\left(\lambda\sqrt{r^2+a^2-2a\,r\cos(\phi-\alpha)}\right) = \sum_{-\infty}^{\infty} J_n(\lambda r)\, J_n(\lambda a)e^{in(\phi-\alpha)}$$

The series that results converges uniformly, and we may integrate term by term. We obtain after these transformations:

$$(5.11)\qquad \underline{\Pi} = \frac{i}{4\pi\omega\epsilon} I\, a\sec\psi\, e^{i\Upsilon z}\sum_{-\infty}^{\infty}\int_0^\infty \frac{\lambda d\lambda}{\sqrt{\lambda^2-k^2}}\, J_n(\lambda r)\, J_n(\lambda a)$$

$$\int_{-\infty}^{\infty} d\alpha\left[\underline{i_r}\sin(\phi-\alpha)\cos\psi + \underline{i_\theta}\cos(\phi-\alpha)\cos\psi + \underline{i_z}\sin\psi\right]e^{in(\phi-\alpha)+ika\sec\psi\alpha-\sqrt{\lambda^2-k^2}\,a\tan\psi|\alpha|}$$

The α integral may be evaluated by elementary means. It is convenient to replace sines and cosines by exponentials. Calling the integral $\underline{U}$, there

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* S. A. Schelkunoff, "Electromagnetic Waves", D. van Nostrand Co., 1943, p. 409

**Stratton, loc. cit., p. 576, Equation 17; Watson loc. cit., p. 416, Equation 4.

***Stratton, loc. cit., p. 373, Equation 8.

results:

(5.12) $$\underline{U} = \int_{-\infty}^{\infty} d\alpha \begin{bmatrix} \underline{i_r} \dfrac{\cos\psi}{2i}\left(e^{i(n+1)(\phi-\alpha)} - e^{i(n-1)(\phi-\alpha)}\right) \\ + \underline{i_\theta} \dfrac{\cos\psi}{2}\left(e^{i(n+1)(\phi-\alpha)} + e^{i(n-1)(\phi-\alpha)}\right) \\ + \underline{i_z} \sin\psi \, e^{in(\phi-\alpha)} \end{bmatrix} e^{ik\,a\sec\psi\alpha - \sqrt{\lambda^2-k^2}\,a\tan\psi|\alpha|}$$

All the integrals may be evaluated as special cases of the general form:

(5.13) $$\int_{-\infty}^{\infty} e^{iu\alpha - v|\alpha|}\, d\alpha = \frac{2v}{u^2+v^2}$$

This gives upon substitution into (5.12)

(5.14)
$$\underline{U} = \sqrt{\lambda^2-k^2}\, a\tan\psi \Bigg[\underline{i_r} \frac{\cos\psi}{i}\left(\frac{e^{i(n+1)\phi}}{(\lambda^2-k^2)a^2\tan^2\psi + (ka\sec\psi-n-1)^2} - \frac{e^{i(n-1)\phi}}{(\lambda^2-k^2)a^2\tan^2\psi + (ka\sec\psi-n+1)^2} \right)$$

$$+ \underline{i_\theta} \cos\psi \left(\frac{e^{i(n+1)\phi}}{(\lambda^2-k^2)a^2\tan^2\psi + (ka\sec\psi-n-1)^2} + \frac{e^{i(n-1)\phi}}{(\lambda^2-k^2)a^2\tan^2\psi + (ka\sec\psi-n+1)^2} \right)$$

$$+ \underline{i_z}\, 2\sin\psi \frac{e^{in\phi}}{(\lambda^2-k^2)a^2\tan^2\psi + (ka\sec\psi-n)^2} \Bigg]$$

Let us factor out $a^2\tan^2\psi$ from the denominator and write $p_n^2 = \left(\frac{n}{a\tan\psi} - k\csc\psi\right)^2 - k^2$. We may note that p_o in our notation is the p of other writers since $k\csc\psi = \gamma$. With this shorthand, we have finally:

(5.15)
$$\underline{U} = \frac{\sqrt{\lambda^2-k^2}}{a\tan\psi} \Bigg[\underline{i_r} \frac{\cos\psi}{i}\left(\frac{e^{i(n+1)\phi}}{\lambda^2 + p_{n+1}^2} - \frac{e^{i(n-1)\phi}}{\lambda^2 + p_{n-1}^2} \right)$$

$$+ \underline{i_\theta} \cos\psi \left(\frac{e^{i(n+1)\phi}}{\lambda^2 + p_{n+1}^2} + \frac{e^{i(n-1)\phi}}{\lambda^2 + p_{n-1}^2} \right)$$

$$+ \underline{i_z}\, 2\sin\psi \frac{e^{in\phi}}{\lambda^2 + p_n^2} \Bigg]$$

Substituting this value for $\underline{U}$ into (5.11) gives:

(5.16)
$$\underline{\Pi} = \frac{i}{4\pi\omega\epsilon} I \csc\psi \; e^{i\gamma z} \sum_{-\infty}^{\infty} \int_0^\infty \lambda d\lambda \, J_n(\lambda r) \, J_n(\lambda a)$$

$$\left[\underline{i_r} \frac{\cos\psi}{i} \left(\frac{e^{i(n+1)\phi}}{\lambda^2 + p_{n+1}^2} - \frac{e^{i(n-1)\phi}}{\lambda^2 + p_{n-1}^2} \right) \right.$$

$$+ \underline{i_\theta} \; \cos\psi \left(\frac{e^{i(n+1)\phi}}{\lambda^2 + p_{n+1}^2} + \frac{e^{i(n-1)\phi}}{\lambda^2 + p_{n-1}^2} \right)$$

$$\left. + \underline{i_z} \; 2 \sin\psi \frac{e^{in\phi}}{\lambda^2 + p_n^2} \right]$$

The various integrals appearing here may all be evaluated as special cases of the general formula*:

(5.17)
$$\int_0^\infty \frac{\lambda d\lambda \, J_n(\lambda r) \, J_n(\lambda a)}{\lambda^2 + p^2} = I_n(pr) \, K_n(pa) \qquad r < a$$

$$I_n(pa) \, K_n(pr) \qquad r > a$$

If p^2 has the negative value $-q^2$, we replace $I_n(pr) \, K_n(pa)$ by $1/2\pi i J_n(qr) H'_n(qa)$. We shall only write down results for $r < a$, since those for $r > a$ can be obtained by interchanging r and a in the formulas for $r < a$. Thus we have:

(5.18)
$$\underline{\Pi} = \frac{i}{4\pi\omega\epsilon} I \csc\psi \, e^{i\gamma z} \sum_{-\infty}^{\infty} \left[\underline{i_r} \frac{\cos\psi}{i} \left(e^{i(n+1)\phi} I_n(p_{n+1} r) K_n(p_{n+1} a) \right. \right.$$

$$\left. - e^{i(n-1)\phi} \, I_n(p_{n-1} r) K_n(p_{n-1} a) \right)$$

$$+ \underline{i_\theta} \cos\psi \left(e^{i(n+1)\phi} I_n(p_{n+1} r) K_n(p_{n+1} a) + e^{i(n-1)\phi} I_n(p_{n-1} r) K_n(p_{n-1} a) \right)$$

$$\left. + \underline{i_z} \; 2 \sin\psi \; e^{in\phi} \, I_n(p_n r) K_n(p_n a) \right]$$

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* Watson, loc. cit. p. 429, Equation 5.

This formula may be simplified by writing $n+1 = n'$ in the first terms of the quantities in parentheses; $n-1 = n'$ in the second terms, and then dropping the primes. The result is:

$$\underline{\Pi} = \frac{i}{4\pi\omega\epsilon} I \csc\psi \; e^{i\gamma z} \sum_{-\infty}^{\infty} e^{in(\theta - \frac{z}{a \tan\psi})} \tag{5.19}$$

$$\Big[\underline{i_r} \frac{\cos\psi}{i} \Big(I_{n-1}(p_n r) \, K_{n-1}(p_n a) - I_{n+1}(p_n r) \, K_{n+1}(p_n a) \Big)$$

$$+ \underline{i_\theta} \cos\psi \Big(I_{n-1}(p_n r) \, K_{n-1}(p_n a) + I_{n+1}(p_n r) \, K_{n+1}(p_n a) \Big)$$

$$+ \underline{i_z} \; 2 \sin\psi \, I_n(p_n r) \, K_n(p_n a) \Big]$$

The convergence of series of this type has been discussed by Phillips[8]. It may easily be shown that if $r \neq a$, the terms of the series decrease geometrically if n is sufficiently large. For $r = a$, the series converges conditionally if $\theta \neq z/a \tan\psi$. In the case $r = a$, $\theta = z/a \tan\psi$, the series diverges. However, this corresponds to the neighborhood of the helical wire itself, and the approximations under which (5.19) was derived are not valid in this region and the divergence does not affect our discussion.

The next step is to obtain the field quantities. These are found from the Hertz vector by the operations*

$$\text{a)} \quad \underline{E} = \nabla (\nabla \cdot \underline{\Pi}) + k^2 \underline{\Pi} \tag{5.20}$$

$$\text{b)} \quad \underline{H} = -i\omega\epsilon \, \nabla \times \underline{\Pi}$$

The formulas for divergence and curl in cylindrical coordinates are:**

$$\text{a)} \quad \nabla \cdot \underline{\Pi} = \frac{1}{r} \frac{\partial}{\partial r} r \Pi_r + \frac{1}{r} \frac{\partial}{\partial \theta} \Pi_\theta + \frac{\partial}{\partial z} \Pi_z \tag{5.21}$$

$$\text{b)} \quad \nabla \times \underline{\Pi} = \underline{i_r} \Big(\frac{1}{r} \frac{\partial}{\partial \theta} \Pi_z - \frac{\partial}{\partial z} \Pi_\theta \Big) + \underline{i_\theta} \Big(\frac{\partial}{\partial z} \Pi_r - \frac{\partial}{\partial r} \Pi_z \Big)$$

$$+ \underline{i_z} \Big(\frac{1}{r} \frac{\partial}{\partial r} r \Pi_\theta - \frac{1}{r} \frac{\partial}{\partial \theta} \Pi_r \Big)$$

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* Stratton, loc. cit., p. 430

**Ibid, p. 51.

The formulas may be simplified by use of the recurrence relations for Bessel functions. The formula:

$$(5.22) \qquad \frac{1}{r}\frac{\partial}{\partial r} r\, I_m(pr) = p\left(I_m'(pr) + \frac{I_m(pr)}{pr}\right)$$

enables us to write out $\nabla \cdot \underline{\Pi}$ as:

$$(5.23) \quad \nabla \cdot \underline{\Pi} = \frac{i}{4\pi\omega\epsilon}\, I\, \csc\psi\, e^{i\gamma z} \sum_{-\infty}^{\infty} e^{in\left(\theta - \frac{z}{a\tan\psi}\right)}$$

$$\left[\frac{\cos\psi}{i}\, p_n \left\{\left(I_{n-1}'(p_n r) + \frac{I_{n-1}(p_n r)}{p_n r}\right) K_{n-1}(p_n a) - \left(I_{n+1}'(p_n r) + \frac{I_{n+1}(p_n r)}{p_n r}\right) K_{n+1}(p_n a)\right)\right\}$$

$$+ \cos\psi\, i p_n \left\{\frac{n I_{n-1}(p_n r)}{p_n r} K_{n-1}(p_n a) + \frac{n I_{n+1}(p_n r)}{p_n r} K_{n+1}(p_n a)\right\}$$

$$+ 2\sin\psi\, i\left(\gamma - \frac{n}{a\tan\psi}\right) I_n(p_n r)\, K_n(p_n a)\Big]$$

Combining the terms in curly brackets gives for the entire bracketed expression:

$$(5.24) \quad \left[\frac{\cos\psi}{i}\, p_n \left\{\left(I_{n-1}'(p_n r) - \frac{(n-1) I_{n-1}(p_n r)}{p_n r}\right) K_{n-1}(p_n a)\right.\right.$$

$$\left. - \left(I_{n+1}'(p_n r) + \frac{(n+1) I_{n+1}(p_n r)}{p_n r}\right) K_{n+1}(p_n a)\right\}$$

$$\left. + 2\sin\psi\, i\left(\gamma - \frac{n}{a\tan\psi}\right) I_n(p_n r)\, K_n(p_n a)\right]$$

The expression in curly brackets here may be simplified by the recurring relations:[*]

$$(5.25) \qquad a) \quad I_\nu'(z) + \frac{\nu}{z} I_\nu(z) = I_{\nu-1}(z)$$

$$b) \quad I_\nu'(z) - \frac{\nu}{z} I_\nu(z) = I_{\nu+1}(z)$$

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* Watson, loc. cit., p. 79, Equations 3 and 4.

When these are inserted we obtain for the curly bracketed expression:

(5.26) $$I_n(p_n r) \left\{ K_{n-1}(p_n a) - K_{n+1}(p_n a) \right\}$$

Another recurrence relation**

(5.27) $$K_{n-1}(p_n a) - K_{n+1}(p_n a) = - \frac{2n}{p_n a} K_n(p_n a)$$

reduces (5.26) to:

(5.28) $$- \frac{2n}{p_n a} I_n(p_n r) K_n(p_n a)$$

We now have as the simplified value of (5.24)

(5.29) $$\left[- \frac{\cos\psi}{i} p_n \cdot \frac{2n}{p_n a} I_n(p_n r) K_n(p_n a) \right.$$

$$\left. + 2 \sin\psi \; i(\gamma - \frac{n}{a \tan\psi}) I_n(p_n r) K_n(p_n a) \right]$$

The first term cancels the term $\frac{n}{a \tan\psi}$ in the factor $\gamma - \frac{n}{a \tan\psi}$,

so the final result is:

(5.30) $$2 \sin\psi \; i\gamma \; I_n(p_n r) K_n(p_n a)$$

(5.31) $$\nabla \cdot \underline{\Pi} = \frac{i}{4\pi\omega\epsilon} I \csc\psi \; e^{i\gamma z} \sum_{-\infty}^{\infty} e^{in(\theta - \frac{z}{a \tan\psi})} \cdot 2ik \; I_n(p_n r) K_n(p_n a)$$

Instead of finding all field components at all points of space, we shall only compute $E_z(r=0)$, which is the relevant quantity for beam interaction. By the prescription of Equations (5.20), we have:

(5.32) $$E_z = \frac{\partial}{\partial z} \nabla \cdot \Pi + k^2 \Pi_z$$

This is equal to:

(5.33) $$E_z = \frac{i}{4\pi\omega\epsilon} I \csc\psi \; e^{i\gamma z} \sum_{-\infty}^{\infty} e^{in(\theta - \frac{z}{a \tan\psi})} I_n(p_n r) K_n(p_n a).$$

$$\left[i(\gamma - \frac{n}{a \tan\psi}) \cdot 2ik + k^2 \cdot 2 \sin\psi \right]$$

** Ibid, p. 79, Equation 1.

At $r = 0$, $I_n(p_n r)$ is zero except for $n = 0$, so the series reduces to the single term:

$$(5.34) \quad E_z(0) = \frac{i}{4\pi\omega\epsilon} I \csc\psi \, e^{i\gamma z} K_0(p_0 a) \, 2 \, (k^2 \sin\psi - k\gamma)$$

$$= \frac{i}{4\pi\omega\epsilon} I \csc\psi \, e^{i\gamma z} K_0(ka \cot\psi) \cdot 2k^2 - \frac{\cos^2\psi}{\sin\psi}$$

$$= - \frac{i}{4\pi\omega\epsilon} I \cdot 2k^2 \cot^2\psi \quad e^{ikz \csc\psi} K_0(ka \cot\psi) \quad .$$

6. Conclusion

We shall now compare the results of this theory with those of previous theories. First, we have obtained a mode which propagates with the phase velocity c along the wire and the phase velocity $c \sin\psi$ along the axis. This has also been obtained in previous theories. However, the restrictions on the validity of the calculations are different. Here the restriction is that $b/a \sec\psi$ should be small. In the helical sheath theory this value of phase velocity is only obtained for ka large.* In the zero thickness wire theory a completely artificial boundary condition is required. Also, we have shown that only two modes exist, propagating with equal velocities in opposite directions. The sheath theory leads to an infinitude of modes, of which only two are propagated. We have not herein considered attenuated modes, which are not generated under our restrictions.

The longitudinal field along the axis is given by equation (5.34). For the purpose of calculating the interaction of the fields with an electron beam, this is the relevant quantity. It may be used as a factor in Pierce's parameter C, which determines the gain of a traveling wave tube.

In our calculations we have obtained the propagation constant correct to lowest order terms in $b/a \sec\psi$. Now it is a consequence of the general theory of guided wave propagation** that the propagation constant of a mode is stationary with respect to small variations from the true field. Hence, the linear term in the series expansion of γ in powers of $b/a \sec\psi$ will vanish, and we may expect

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*Chu and Jackson, loc. cit., Equation 38 ff.

**Schelkunoff, loc. cit., p. 384, J. Schwinger, unpublished notes.

that a more accurate calculation will yield a γ of the form:

(6.1) $$\gamma = k \csc\psi \left[1 + c_1 (b/a \sec\psi)^2\right]$$

The numerical coefficient c_1 will probably be on the order of magnitude of $\frac{1}{4}$ to $\frac{1}{2}$.

The major results we have obtained are the propagation constant and the presence of only one mode. These have been confirmed by experiment.[10] Because of the difficulty of calculating the fields by use of the circulating wave expansion, Section 5, no attempt has been made to compare the field strength within the helix with experiment.

We may sum up as follows. The method herein presented obtains all the results of previous theories of a cold helix. Moreover, it obtains them in a consistent manner without employing any artificial boundary conditions, but directly from Maxwell's equations and the standard perfect conductivity boundary conditions.

As far as mathematical interest is concerned, the major point is the use of a non-orthogonal coordinate system. Despite the non-separability of the differential equations, it is still possible to obtain all desired results. The method may be extended to treat the higher approximations and obtain the attenuated modes.

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